

Ovoidal fibrations in $PG(3, q)$, q even

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Abstract

We prove that, given a partition of the point-set of $PG(3, q)$, $q = 2^n > 2$, by ovoids $\{\theta_i\}_{i=0}^q$ of $PG(3, q)$ and a line ℓ of $PG(3, q)$, not tangent to θ_0 if ℓ^\perp denotes the polar of ℓ relative to the symplectic form on $PG(3, q)$ whose isotropic lines are the tangent lines to θ_0 , then ℓ and ℓ^\perp are tangent to distinct ovoids θ_j, θ_k , both distinct from θ_0 (Theorem 1.2). This uses the fact that the radical of the linear code generated by the dual duals $\ell \cup \ell^\perp$ of the hyperbolic quadrics, with ℓ and ℓ^\perp as above, is of codimension 1 (Theorem 1.4)

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1 Introduction and Statement of the Result

An *ovoid* in $PG(3, q)$ is a set of $q^2 + 1$ points, no three collinear. Elliptic quadrics and the Suzuki-Tits ovoids ([24], See [13], 16.4),

which exist, if and only if, n is odd, are the only known ovoids in $PG(3, q)$ and these are the only ovoids if $n \leq 6$ [17]. Classification of ovoids in $PG(3, q)$ is a fundamental problem in Incidence Geometry. We mention in passing that the elliptic quadrics are the only ovoid in $PG(3, q)$ if q is odd (independently due to Barlotti [4] and Panella [18], see [17], Theorem 2.1, p.178). We recall that a *general linear complex* in $PG(3, q)$ is the set of all absolute lines with respect to a symplectic polarity of $PG(3, q)$ (that is, a polarity for which all points are absolute). If P is the set of all points of $PG(3, q)$ and L is a general linear complex in $PG(3, q)$, then the incidence system (P, L) , which we denote by $W(q)$, is a generalized quadrangle of order (q, q) (see [19], p.37). Since the collineation group $PGL(4, q)$ acts by conjugation transitively on the set of all symplectic polarities of $PG(3, q)$, $W(q)$ is uniquely defined, up to a collineation of $PG(3, q)$. An ovoid of $W(q)$ is a set \mathcal{O} of $q^2 + 1$ points, pairwise noncollinear in $W(q)$. Then, any line l of $PG(3, q)$ meets \mathcal{O} in a point if it is a line of $W(q)$ and in zero or two points if it is not a line of $W(q)$ as shown by a simple count. Thus any ovoid of $W(q)$ is an ovoid of $PG(3, q)$. The following observation, due to Segre [21], proves the converse: let \mathcal{O} be an ovoid of $PG(3, q)$, q even, and $W(\mathcal{O})$ be the incidence system whose points are the points of $PG(3, q)$ and lines are the tangent lines of \mathcal{O} . For each $x \in \mathcal{O}$, the union of the set of the $q + 1$ tangent lines to \mathcal{O} through x is a plane π_x . The correspondence $x \mapsto \pi_x$ defines a symplectic polarity (= null polarity) of $PG(3, q)$ [9] whose absolute lines are precisely the lines of $W(\mathcal{O})$. Thus, $W(\mathcal{O}) = g(W(q))$ for some $g \in PGL(4, q)$ and the ovoid \mathcal{O} of $PG(3, q)$ is an ovoid of (a unique copy of) $W(q)$. Hence, the classification of ovoids of $PG(3, q)$, q even, is equivalent to the classification of ovoids in $W(q)$.

It is easy to see that any three mutually skew lines in $PG(3, q)$ have exactly $q + 1$ transversals (lines meeting each of the three given lines). Such a set of $q + 1$ transversals is called a *regulus* in $PG(3, q)$. The transversals to the lines in any regulus form another regulus, called the *opposite* of the given regulus. Thus, any three mutually skew lines of $PG(3, q)$ are in a unique regulus. A *spread* of $PG(3, q)$ is a set of $q^2 + 1$ mutually skew lines. A spread \mathcal{S} is said to be *regular* (see [9], 5.1) if the unique regulus containing three lines in \mathcal{S} is contained in \mathcal{S} .

An *ovoidal fibration* $\mathcal{F} = \{\theta_i\}_{i=0}^q$ of $PG(3, q)$ is a set of $q + 1$ ovoids partitioning the point set of $PG(3, q)$. We recall that a Singer group in $PGL(4, q)$ is a cyclic group of order $(q^2 + 1)(q + 1)$ acting transitively

on the point-set of $P(3, q)$. Let S be a Singer group in $PGL(4, q)$, and T, K be its unique subgroups of orders $q^2 + 1$ and $q + 1$, respectively. Then, the point T -orbits $\{E_i\}_{i=0}^q$ in $PG(3, q)$ is an ovoidal fibration by elliptic quadrics (see [1], Lemma 2, p.141); also [10], Theorem 3 and [7]); the set \mathcal{S} of common tangent lines to the E_i 's is a regular spread in $PG(3, q)$; T acts regularly on the set \mathcal{S} and K acts regularly on each member of \mathcal{S} ([12]; see [1], Lemma 2, p.141). More generally, if θ is an ovoid of $PG(3, q)$ and \mathcal{S} is contained in the general linear complex $\mathcal{L} = \mathcal{L}(\theta)$ consisting of all tangent lines to θ (see, Segre [21]) and if K is the group of collineations fixing each line in \mathcal{S} , then K is a subgroup of order $q + 1$ contained in a Singer subgroup of $PGL(4, q)$ (see [12]) and the set $\{x(\theta) : x \in K\}$ of ovoids in $PG(3, q)$ is an ovoidal fibration of $PG(3, q)$ (see [5], Theorem 3.1, p.161). Further, θ is the only ovoid of the symplectic generalized quadrangle $(P, \mathcal{L}) \simeq W(q)$, where P is the point-set of $PG(3, q)$ (see [1], Lemma 2, p.141).

Let θ be an ovoid of $W(q)$ and ' \perp ' denote the orthogonality relative to the nondegenerate symplectic form on $PG(3, q)$ whose isotropic lines are the tangent lines to θ . Since the symplectic polarity defined by θ maps an external line to θ to a secant line to θ and vice-versa, if m is a line of $PG(3, q)$ which is not tangent to θ , then one of the lines m, m^\perp shares two points with θ and the other is disjoint from it ([13], Corollary 1 of Theorem 16.1.8). Consequently, θ is an ovoid of $PG(3, q)$ also. Thus, the classification of ovoids in $PG(3, q)$ and in $W(q)$ are equivalent problems. We do not know any examples of ovoidal fibration containing a pair of projectively nonequivalent ovoids (see ([6]) for a discussion of the case $q = 8$); and of any examples of fibrations consisting of projectively equivalent ovoids, but with no transitive action of a subgroup of $PGL(4, q)$ on its constituent ovoids.

The subset $m \cup m^\perp$ of P , with m, m^\perp as above, is called a *dual grid* of $W(q)$. With the lines of $W(q)$ incident with it as 'lines', $m \cup m^\perp$ is a $(1, q)$ -subgeneralized quadrangle of $W(q)$. We denote by \mathbb{D} the set of dual grids in $W(q)$.

Proposition 1.1 *Let $\{\theta_i\}_{i=0}^q$ be an ovoidal fibration of $PG(3, q)$ and \mathcal{L}_i denote the general linear complex consisting of all tangent lines to $\theta_i, 0 \leq i \leq q$. Then, the following hold:*

- (i) *The set \mathcal{S} of common tangent lines to the ovoids θ_i is a regular spread in $PG(3, q)$. Further, $\{\mathcal{L}_i\}_{i=0}^q$ are the only general linear*

complexes in $PG(3, q)$ containing \mathcal{S} ; and $\mathcal{L}_i \cap \mathcal{L}_j = \mathcal{S}$ for all $0 \leq i \neq j \leq q$.

- (ii) Each line ℓ of $PG(3, q)$ not in \mathcal{S} is tangent to a unique ovoid θ_i , secant to $q/2$ ovoids θ_j , $j \neq i$, and is disjoint from each of the remaining $q/2$ ovoids θ_k of the fibration.

Our object in this note is to prove

Theorem 1.2 *Let $\{\theta_i\}_{i=0}^q$ be an ovoidal fibration of $PG(3, q)$, $W(q)$ be the generalized quadrangle whose line set is the set $\mathcal{L}(\theta_0)$ of tangent lines to θ_0 and m be a line of $PG(3, q)$ not in $\mathcal{L}(\theta_0)$. Then, m and its perp m^\perp in $W(q)$ are tangent to distinct ovoids θ_i , each distinct from θ_0 .*

An intrinsic description of the ovoids θ_i the lines m and m^\perp are tangent to may be interesting. We would like to view this note as a contribution towards understanding the packings of $PG(3, q)$ by ovoids.

Our proof of Theorem 1.2 uses a property of the binary code we now define. Let G denote the projective symplectic subgroup $PSp(4, q)$ of $PGL(4, q)$ defined by $W(q)$. Let \mathbb{F}_2^P and $\mathbb{F}_2^\mathbb{D}$ denote the \mathbb{F}_2G -permutation modules on P and \mathbb{D} , respectively, and η denote the \mathbb{F}_2G -module homomorphism taking $m \cup m^\perp \in \mathbb{D}$ to $\sum_{x \in m \cup m^\perp} x \in \mathbb{F}_2^P$. We identify the characteristic function of a subset of P , considered as an element of \mathbb{F}_2^P , with the subset itself. Let \mathcal{C} and \mathcal{D} denote the \mathbb{F}_2G -submodules of \mathbb{F}_2^P whose generators are, respectively, the lines of $W(q)$ and the dual grids of $W(q)$. Then, $\eta(\mathbb{F}_2^\mathbb{D}) = \mathcal{D}$. Since any line of $W(q)$ meets a dual grid in zero or two points, \mathcal{D} is contained in the dual code \mathcal{C}^\perp of \mathcal{C} . Further, it is generated by words of \mathcal{C}^\perp of minimum weight ([2], Theorem 1.4).

Corollary 1.3 $\mathcal{D} \subseteq \mathcal{C}^\perp_+$.

We need the following

Theorem 1.4 *The \mathbb{F}_2G -radical U of \mathcal{D} is of codimension one in \mathcal{D} . Consequently, the sum of two dual grids of $W(q)$ is in the radical U .*

Lemma 1.5 *The group G contains a unique conjugacy class of subgroups T of order $q^2 + 1$ and T is cyclic. Let $\{E_i\}_{i=0}^q$ be the point T -orbits in $PG(3, q)$ and \mathcal{S} denote the set of all common tangent lines*

to E_i 's. Let ℓ be a line of $PG(3, q)$. Then, $\sum_{t \in T} t(\ell) \in \mathbb{F}_2^P$ is P (i.e., the 'all-one' vector) or the unique ovoid E_i the line ℓ is tangent to, according as $\ell \in \mathcal{S}$ or $\ell \notin \mathcal{S}$.

2 Preliminaries

Let $V(q)$ be the vector space of dimension four over \mathbb{F}_q ; \hbar be a non-degenerate symplectic bilinear form on it; and $W(q)$ be the incidence system with the set $P = PG(3, q)$ of all one dimensional subspaces of $V(q)$ as its *point-set*, the set L of all two dimensional subspaces of $V(q)$ which are isotropic with respect to \hbar as its *line-set* and symmetrized inclusion as the *incidence*. Then, $W(q)$ is a regular generalized quadrangle of order q ([19], p.37) and the symplectic group G defined by \hbar acts as incidence preserving permutations on the sets P and L .

Let k be an algebraically closed extension field of \mathbb{F}_q . Let $N = \{0, 1, \dots, 2n - 1\}$. Addition in N is always taken modulo $2n$. Let \mathcal{N} denote the set of all subsets I of N containing no consecutive elements. Let $V = V(q) \otimes k$. The natural extension of the symplectic form \hbar to V defined above is also denoted by \hbar . Then, G is the subgroup of the algebraic group $Sp(V) \simeq Sp(4, k)$ fixed by the n -th power of the Frobenius map σ (which is the algebraic group endomorphism of $GL(4, k)$ raising each entry of a matrix to its 2^{nd} -power). It is well known that $Sp(V)$ has an algebraic group endomorphism τ with $\tau^2 = \sigma$ ([23], Theorem 28, p.146). For any non-negative integer i , we denote by V_i the $Sp(V)$ -module whose k -vector space structure is the same as that of V and an element g of $Sp(V)$ acts on V_i as $\tau^i(g)$ would act on V . For $I \subseteq N$, let V_I denote the kG -module $\otimes_{i \in I} V_i$ (with $V_\emptyset = k$). Then, by Steinberg's tensor product Theorem ([22], §11), $\{V_I : I \subseteq N\}$ is a complete set of inequivalent simple kG -modules. For a kG -module M , we denote by $[M : V_I]$ the multiplicity of V_I in a composition series of M . We denote by $\text{rad}(M)$ the *radical* of M (that is, the smallest submodule of M with semisimple quotient). We refer to $M/\text{rad}(M)$ as the *head* of M .

We now describe a graph automorphism τ of $Sp(V)$, following ([11], pp.58-60). (The argument presented in loc. cit. constructs a graph automorphism for $G = Sp(V(q))$, however the arguments are valid for $Sp(V)$ also.) Let $\{e_1, e_2, e_3, e_4\}$ be an ordered basis of V and Q denote the nondegenerate quadratic form on the exterior square $\Lambda^2 V$

of V defined by

$$Q(\Sigma_{1 \leq i < j \leq 4} \lambda_{ij} e_i \wedge e_j) = \lambda_{12} \lambda_{34} + \lambda_{13} \lambda_{24} + \lambda_{14} \lambda_{23}$$

(whose zero set in $P(\Lambda^2 V)$ is the well-known *Klein quadric*). Let β denote the polarization of Q and $\gamma = e_1 \wedge e_4 + e_2 \wedge e_3 \in \Lambda^2 V$. Then, the restriction of Q to the hyperplane $U = \{x \in \Lambda^2 V : \beta(x, \gamma) = 0\}$ of $\Lambda^2 V$ is a nondegenerate quadratic form; and the restriction of β to U is an alternating form with radical $k\gamma$. The alternating form $\bar{\beta}$ induced by β on $\bar{U} = \frac{U}{k\gamma}$ is nondegenerate. So the symplectic space $(\bar{U}, \bar{\beta})$ is isometric to (V, \hbar) . Let $\bar{p} : \bar{U} \rightarrow V$ be the isometric isomorphism induced by the linear map $p : U \rightarrow V$ defined by

$$\begin{aligned} p(e_1 \wedge e_2) &= e_1, \quad p(e_1 \wedge e_3) = e_2, \quad p(e_2 \wedge e_4) = e_3, \\ p(e_3 \wedge e_4) &= e_4, \quad p(\gamma) = 0. \end{aligned}$$

Then the map taking $g \in Sp(V)$ to $\bar{p}(\overline{\wedge^2(g)}) \bar{p}^{-1} \in Sp(V)$ is a graph automorphism τ of $Sp(V)$ which, on restriction to G , gives a graph automorphism of G .

We use the following results:

Lemma 2.1 ([8], Corollary 7.11, p.148) *Let K be a field of characteristic $p > 0$, let E be a field extension of K and let X be a finite group. Then, for each simple KX -module M , $M^E = M \otimes_K E$ is a direct sum of simple EX -modules, no two of which are isomorphic.*

Lemma 2.2 ([20], Corollary 6) *Let $K, K' \in \mathcal{N}$ be distinct and f_{\hbar} be a nondegenerate quadratic form on $V(q)$ of index 2 which polarizes to \hbar . Then, V_K and $V_{K'}$ are semi-simple $k\Omega(f_{\hbar})$ -module with no irreducible factors in common.*

3 Proofs

Proof of Theorem 1.4: Let $g \in \mathbb{D}$ and $L = \text{Stab}_G(g)$. We view $\mathbb{F}_2^{\mathbb{D}}$ as the induced module of the trivial $\mathbb{F}_2 G$ -module \mathbb{F}_2 , and write $\text{Ind}_L^G(\mathbb{F}_2) = \mathbb{F}_2^{\mathbb{D}}$. Then,

$$[\mathbb{F}_2^{\mathbb{D}} / \text{Rad}(\mathbb{F}_2^{\mathbb{D}}) : \mathbb{F}_2] = 1$$

because, by Frobenius reciprocity ([16], p.689),

$$\dim_{\mathbb{F}_2}(Hom_{\mathbb{F}_2 G}(Ind_L^G(\mathbb{F}_2), \mathbb{F}_2)) = \dim_{\mathbb{F}_2}(Hom_{\mathbb{F}_2 L}(\mathbb{F}_2, \mathbb{F}_2)) = 1.$$

This shows that the trivial module \mathbb{F}_2 is a summand of the head (that is, the largest semi-simple quotient) of the $\mathbb{F}_2 G$ -module $\mathbb{F}_2^{\mathbb{D}}$. Let \mathcal{H} denote the set of all hyperbolic quadrics of $W(q)$ and f_h be a non-degenerate quadratic form on $V(q)$ of index 2 which polarizes to \bar{h} . Then, the variety defined by f_h is a member of \mathcal{H} . Let $O(f_h) \subseteq G$ denote the orthogonal group of f_h and $\Omega(f_h)$ denote its commutator subgroup. Let I be a nonempty subset of N . Then by Lemma 2.2,

$$Hom_{k\Omega(f_h)}(k, V_I) \cong Hom_{k\Omega(f_h)}(V_{I_e}, V_{I_o}) = \{0\}.$$

This proves that V_I has no nonzero fixed points for the action of $O(f_h)$.

Since G acts transitively on \mathcal{H} and since the automorphism τ maps the stabilizer of a hyperbolic quadric in $W(q)$ to the stabilizer of a dual grid in $W(q)$ and vice-versa,

$$\dim_k(Hom_{kG}(Ind_L^G(k), V_I)) = \dim_k(Hom_{kG}(Ind_{O(f_h)}^G(k), V_I)).$$

Again using Frobenius reciprocity, we have

$$\begin{aligned} \dim_k(Hom_{kG}(Ind_L^G(k), V_I)) &= \dim_k(Hom_{kG}(Ind_{O(f_h)}^G(k), V_I)) \\ &= \dim_k(Hom_{kO(f_h)}(k, V_I)) \\ &\leq \dim_k(Hom_{k\Omega(f_h)}(k, V_I)) = 0 \end{aligned}$$

This proves that the head of the kG -module $k^{\mathbb{D}} \cong Ind_L^G(k)$ is k . Now the Lemma 2.1 completes the proof of Theorem 1.4.

Proof of Proposition 1.1: The statement (i) follows from ([3], Theorem 2.2) and ([5], Lemma 2.4). Since $|\mathcal{L}_i| = (q^2 + 1)(q + 1)$ and $\mathcal{L}_i \cap \mathcal{L}_j = \mathcal{S}$ which has $q^2 + 1$ elements for all i, j , $i \neq j$ (see [3], Corollary 3.3), each line ℓ of $PG(3, q)$ not in \mathcal{S} is in \mathcal{L}_i for a unique i and $|\ell \cap \theta_j| \in \{0, 2\}$ for all $j \neq i$. Since $\{\theta_i\}$ partitions $PG(3, q)$, (ii) follows.

Proof of Lemma 1.5: The first statement follows from ([11], Lemma 3); the second from ([1], Lemma 2, p.141); and the third follows by Proposition 1.1(ii) and the regular action of T on each E_i (see [1], Lemma 4, p.142).

Proof of Theorem 1.2: Let \mathcal{S} be the set of common tangent lines to the θ_i 's. Then, \mathcal{S} is a regular spread in $PG(3, q)$ by Proposition 1.1

(i). Let H be the subgroup of $PGL(4, q)$ fixing each line in \mathcal{S} . Then H is contained in a Singer subgroup K of $PGL(4, q)$ (see [12]). Consider the unique subgroup T of K of order $q^2 + 1$. The point-orbits of T are elliptic ovoids E_i , forming an ovoidal fibration of $PG(3, q)$ and \mathcal{S} is also the set of common tangents to E_i 's (See [1], Lemma 2, p.141). Let \mathcal{L}_i be the set of all tangent lines to θ_i and $W(q) = W(\mathcal{L}_0)$. Since \mathcal{S} is contained in exactly $q + 1$ general linear complexes (see Proposition 1.1 (i)), we may assume that \mathcal{L}_i is also the set of all tangent lines to E_i . Thus θ_0 and E_0 are ovoids of $W(q) = W(L_0)$ and T is a subgroup of $G (= Sp(4, q)$ defined by $W(q)$). Now, assume that $m \cup m^\perp$ is a dual grid of $W(q)$ such that m and m^\perp are tangents to θ_i for some $i \neq 0$. Then, both are tangents to E_i also. Since $m, m^\perp \notin \mathcal{S}$, by Lemma 1.5,

$$\sum_{t \in T} t(m + m^\perp) = E_i + E_i = 0.$$

On the other hand, since for each $t \in T$, $t(m + m^\perp) \in \mathcal{D}$ and the codimension of U in \mathcal{D} is one, $\sum_{t \in T} t(m + m^\perp) = m + m^\perp \pmod{U}$. Since G is transitive on \mathbb{D} , U can not contain any dual grid of $W(q)$. This completes the proof of the theorem.

Proof of Corollary 1.3: Let T be a cyclic subgroup of order $q^2 + 1$ contained in the stabilizer in G of an elliptic ovoid θ_0 of $W(q)$. Let $\{\theta_i\}_{i=0}^q$ be the point T -orbits in $PG(3, q)$ and \mathcal{S} denote the set of all common tangent lines to θ_i 's (See [1], Lemma 2, p.141). Consider the \mathbb{F}_2 -linear map $\sigma : \mathbb{F}_2^P \longrightarrow \mathbb{F}_2^P$ defined by $\sigma(w) = \sum_{t \in T} t(w)$. As we noted earlier, $\mathcal{D} \subseteq \mathcal{C}^\perp$. Assume that $\mathcal{C}^\perp = \mathcal{D}$. Then $\mathcal{C} = (\mathcal{C}^\perp)^\perp = \mathcal{D}^\perp \supseteq \mathcal{D}$, as any two dual grids of $W(q)$ intersect in zero or two points. Now, $\sigma(\mathcal{C}) = \{\emptyset, P, \theta_0, P \setminus \theta_0\}$ (see Lemma 1.5); however if $m \cup m^\perp$ is a dual grid of $W(q)$, then by Lemma 1.5 and Theorem 1.2 $\sigma(m + m^\perp) = \theta_i + \theta_j$ for distinct i and j ($i > 0, j > 0$), a contradiction.

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